Lecture 02
12.2/12.3 Vector algebra and the dot product

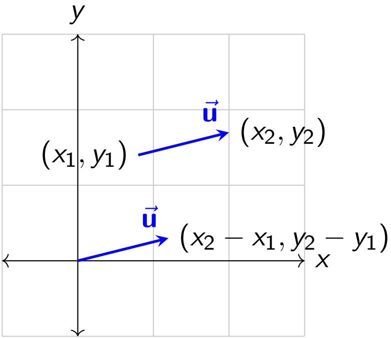
Jeremiah Southwick

January 16, 2019

Vectors

The direction is where the arrow points and the length is how long the arrow is.

A vector models any application where force is involved: velocity, displacement, work, etc.



Since it doesn't matter where we draw a vector, we will usually place the initial point at the origin. This is called *standard position*.

Standard Position

Definition

If a vector $\vec{\mathbf{v}}$ *goes from* (x_1, y_1, z_1) *to* (x_2, y_2, z_2)*, then the same vector in standard position goes from* $(0, 0, 0)$ *to* $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$*. In this case we write*

$$\vec{\mathbf{v}} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$

$$\langle u_1, u_2, u_3 \rangle = \langle v_1, v_2, v_3 \rangle \;\Leftrightarrow\; u_1 = v_1, u_2 = v_2, \text{ and } u_3 = v_3$$

Length/Magnitude

The length of a vector is simply the distance from its initial point to its terminal point.

Definition

If $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$, *then the length of* $\vec{\mathbf{v}}$ *is*

$$\|\vec{\mathbf{v}}\| = |\vec{\mathbf{v}}| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

Example

The length of $\langle 7, 3, -2 \rangle$ *is* $\sqrt{49 + 9 + 4} = \sqrt{62}$.

Vector Algebra

Given two vectors, we will now describe different ways to combine them, which will turn the set of vectors into something we can manipulate algebraically.

Vector Algebra

Given two vectors, we will now describe different ways to combine them, which will turn the set of vectors into something we can manipulate algebraically.

Definition

If $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$, $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$, *and* $k \in \mathbb{R}$, *then we have the following operations:*
Vector addition:

$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$$

Vector Algebra

Given two vectors, we will now describe different ways to combine them, which will turn the set of vectors into something we can manipulate algebraically.

Definition

If $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$, $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$, *and* $k \in \mathbb{R}$, *then we have the following operations:*
Vector addition:

$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$$

Scalar multiplication:

$$k\vec{\mathbf{u}} = \langle ku_1, ku_2, ku_3 \rangle.$$

Vector Algebra

Given two vectors, we will now describe different ways to combine them, which will turn the set of vectors into something we can manipulate algebraically.

Definition

If $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$, $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$, *and* $k \in \mathbb{R}$, *then we have the following operations:*
Vector addition:

$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$$

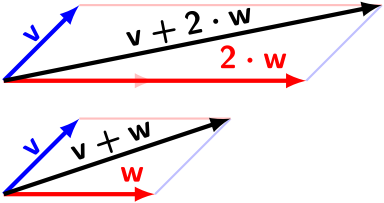
Scalar multiplication:

$$k\vec{\mathbf{u}} = \langle ku_1, ku_2, ku_3 \rangle.$$

This means we can add two vectors and multiply vectors by numbers.

Vector Algebra

These operations have a geometric meaning.



Vector addition corresponds to following first one vector and then the other to the resulting location. Scalar multiplication corresponds to stretching/shrinking a vector without changing its direction.

Vector Algebra

Example

Let $\vec{\mathbf{v}} = \langle 2, 4 \rangle$ *and* $\vec{\mathbf{u}} = \langle -4, 6 \rangle$. *Find the component form of* $\frac{1}{2}(\vec{\mathbf{v}} + \vec{\mathbf{u}})$.

Vector Algebra

Example

Let $\vec{\mathbf{v}} = \langle 2, 4\rangle$ *and* $\vec{\mathbf{u}} = \langle -4, 6\rangle$. *Find the component form of* $\frac{1}{2}(\vec{\mathbf{v}} + \vec{\mathbf{u}})$.

$$\frac{1}{2}(\vec{\mathbf{v}} + \vec{\mathbf{u}}) = \frac{1}{2}(\langle 2, 4\rangle + \langle -4, 6\rangle) = \frac{1}{2}\langle -2, 10\rangle = \langle -1, 5\rangle.$$

Properties (page 712)

1. $\vec{\mathbf{u}} + \vec{\mathbf{v}} = \vec{\mathbf{v}} + \vec{\mathbf{u}}$
2. $(\vec{\mathbf{u}} + \vec{\mathbf{v}}) + \vec{\mathbf{w}} = \vec{\mathbf{u}} + (\vec{\mathbf{v}} + \vec{\mathbf{w}})$
3. $\vec{\mathbf{u}} + \vec{\mathbf{0}} = \vec{\mathbf{u}}$
4. $\vec{\mathbf{u}} + (-\vec{\mathbf{u}}) = \vec{\mathbf{0}}$
5. $0\vec{\mathbf{u}} = \vec{\mathbf{0}}$
6. $1\vec{\mathbf{u}} = \vec{\mathbf{u}}$
7. $a(b\vec{\mathbf{u}}) = (ab)\vec{\mathbf{u}}$
8. $a(\vec{\mathbf{u}} + \vec{\mathbf{v}}) = a\vec{\mathbf{u}} + a\vec{\mathbf{v}}$
9. $(a + b)\vec{\mathbf{u}} = a\vec{\mathbf{u}} + b\vec{\mathbf{u}}$

Properties

3. $\vec{\mathbf{u}} + \vec{\mathbf{0}} = \vec{\mathbf{u}}$

Properties

3. $\vec{\mathbf{u}} + \vec{\mathbf{0}} = \vec{\mathbf{u}}$
Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$.

$\vec{\mathbf{u}} + \vec{\mathbf{0}}$	Left hand side of equation
$= \langle u_1, u_2, u_3 \rangle + \langle 0, 0, 0 \rangle$	Definition of vectors
$= \langle u_1 + 0, u_2 + 0, u_3 + 0 \rangle$	Vector addition
$= \langle u_1, u_2, u_3 \rangle$	0 additive identity
$= \vec{\mathbf{u}}$	Right hand side of equation

Unit Vectors

We can change the length of a vector with scalar multiplication, so we are interested in vectors that have length 1. These vectors have a specific name.

Unit Vectors

We can change the length of a vector with scalar multiplication, so we are interested in vectors that have length 1. These vectors have a specific name.

Definition

A vector $\vec{\mathbf{v}}$ *is a* <u>*unit vector*</u> *if its length is 1.*

Unit Vectors

We can change the length of a vector with scalar multiplication, so we are interested in vectors that have length 1. These vectors have a specific name.

Definition

A vector $\vec{\mathbf{v}}$ is a unit vector if its length is 1.

Example

$\vec{\mathbf{v}} = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$ *is a unit vector*

Unit Vectors

We can change the length of a vector with scalar multiplication, so we are interested in vectors that have length 1. These vectors have a specific name.

Definition

A vector $\vec{\mathbf{v}}$ is a unit vector if its length is 1.

Example

$\vec{\mathbf{v}} = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$ *is a unit vector because*

$$\|\vec{\mathbf{v}}\| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = 1.$$

Standard Unit Vectors

In 3D, we have special unit vectors which point in the direction of the $x-$, $y-$ and z-axes.

Standard Unit Vectors

In 3D, we have special unit vectors which point in the direction of the $x-$, $y-$ and z-axes.

Definition

The standard unit vectors are $\vec{\mathbf{i}} = \langle 1, 0, 0 \rangle, \vec{\mathbf{j}} = \langle 0, 1, 0 \rangle$, *and* $\vec{\mathbf{k}} = \langle 0, 0, 1 \rangle$.

Standard Unit Vectors

In 3D, we have special unit vectors which point in the direction of the $x-, y-$ and z-axes.

Definition

The standard unit vectors are $\vec{\mathbf{i}} = \langle 1, 0, 0\rangle, \vec{\mathbf{j}} = \langle 0, 1, 0\rangle$*, and* $\vec{\mathbf{k}} = \langle 0, 0, 1\rangle$.

We can break up any vector as
$\vec{\mathbf{v}} = \langle v_1, v_2, v_3\rangle = \langle v_1, 0, 0\rangle + \langle 0, v_2, 0\rangle + \langle 0, 0, v_3\rangle = v_1\vec{\mathbf{i}} + v_2\vec{\mathbf{j}} + v_3\vec{\mathbf{k}}$.

Vectors

Note that we can always change a (nonzero) vector to a unit vector in the same direction by dividing by its length, that is, $\left(\frac{1}{\|\vec{\mathbf{v}}\|}\right)\vec{\mathbf{v}}$ is a unit vector in the direction of $\vec{\mathbf{v}}$.

Vectors

Note that we can always change a (nonzero) vector to a unit vector in the same direction by dividing by its length, that is, $\left(\frac{1}{\|\vec{\mathbf{v}}\|}\right)\vec{\mathbf{v}}$ is a unit vector in the direction of $\vec{\mathbf{v}}$.

Example

Find a unit vector in the direction of $\vec{\mathbf{u}} = 2\vec{\mathbf{i}} + 3\vec{\mathbf{j}} - 4\vec{\mathbf{k}}$.

Vectors

Note that we can always change a (nonzero) vector to a unit vector in the same direction by dividing by its length, that is, $\left(\frac{1}{\|\vec{\mathbf{v}}\|}\right)\vec{\mathbf{v}}$ is a unit vector in the direction of $\vec{\mathbf{v}}$.

Example

Find a unit vector in the direction of $\vec{\mathbf{u}} = 2\vec{\mathbf{i}} + 3\vec{\mathbf{j}} - 4\vec{\mathbf{k}}$.

$\|\vec{\mathbf{u}}\| = \sqrt{2^2 + 3^2 + (-4)^2} = \sqrt{29}$. Thus the vector

$$\frac{\vec{\mathbf{u}}}{\|\vec{\mathbf{u}}\|} = \frac{2}{\sqrt{29}}\vec{\mathbf{i}} + \frac{3}{\sqrt{29}}\vec{\mathbf{j}} - \frac{4}{\sqrt{29}}\vec{\mathbf{k}}$$

is a unit vector in the direction of $\vec{\mathbf{u}}$.

§12.3 The dot product

§12.3 The dot product

The *dot product* of two vectors gives us geometric information about the angle between the vectors.

Definition

Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ *and* $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$. *Then*

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

is the dot product of $\vec{\mathbf{u}}$ *and* $\vec{\mathbf{v}}$.

§12.3 The dot product

The *dot product* of two vectors gives us geometric information about the angle between the vectors.

Definition

Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ *and* $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$. *Then*

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

is the dot product of $\vec{\mathbf{u}}$ *and* $\vec{\mathbf{v}}$.

The dot product of two vectors is a number, not a vector.

Dot product example

Example

$\langle 1, -2, -1\rangle \cdot \langle -6, 2, -3\rangle =$

Dot product example

Example

$\langle 1, -2, -1\rangle \cdot \langle -6, 2, -3\rangle = (1)(-6) + (-2)(2) + (-1)(-3) = -7.$

Angles

The dot product tells us something about the angles between two vectors.

Angles

The dot product tells us something about the angles between two vectors.
Let $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ be nonzero vectors. If θ is the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, then

$$\theta = \arccos\left(\frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|\,\|\vec{\mathbf{v}}\|}\right).$$

Angles

The dot product tells us something about the angles between two vectors.
Let $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ be nonzero vectors. If θ is the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, then

$$\theta = \arccos\left(\frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|}\right).$$

Two other ways of saying this are

$$\cos(\theta) = \frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|} \qquad \text{or}$$

Angles

The dot product tells us something about the angles between two vectors.

Let $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ be nonzero vectors. If θ is the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, then

$$\theta = \arccos\left(\frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\| \|\vec{\mathbf{v}}\|}\right).$$

Two other ways of saying this are

$$\cos(\theta) = \frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\| \|\vec{\mathbf{v}}\|} \qquad \text{or} \qquad \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \|\vec{\mathbf{u}}\| \|\vec{\mathbf{v}}\| \cos(\theta).$$

Angles

The dot product tells us something about the angles between two vectors.
Let $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ be nonzero vectors. If θ is the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, then

$$\theta = \arccos\left(\frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|}\right).$$

Two other ways of saying this are

$$\cos(\theta) = \frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|} \qquad \text{or} \qquad \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| \cos(\theta).$$

Can be proven using the law of Cosines (page 719).

Angle example

$$\theta = \arccos\left(\frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\| \|\vec{\mathbf{v}}\|}\right)$$

Example

Find the angle between $\vec{\mathbf{u}} = \vec{\mathbf{i}} - 2\vec{\mathbf{j}} - 2\vec{\mathbf{k}}$ *and* $\vec{\mathbf{v}} = 6\vec{\mathbf{i}} + 3\vec{\mathbf{j}} + 2\vec{\mathbf{k}}$.

Angle example

$$\theta = \arccos\left(\frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|}\right)$$

Example

Find the angle between $\vec{\mathbf{u}} = \vec{\mathbf{i}} - 2\vec{\mathbf{j}} - 2\vec{\mathbf{k}}$ *and* $\vec{\mathbf{v}} = 6\vec{\mathbf{i}} + 3\vec{\mathbf{j}} + 2\vec{\mathbf{k}}$.

We have $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 1(6) - 2(3) - 2(2) = -4$,
$\|\vec{\mathbf{u}}\| = \sqrt{1^2 + (-2)^2 + (-2)^2} = 3$, and $\|\vec{\mathbf{v}}\| = \sqrt{6^2 + 3^2 + 2^2} = 7$.
Thus,

$$\theta = \arccos\left(-\frac{4}{21}\right) \approx 1.762 \text{ rad} \approx 100.98^\circ.$$

Orthogonal Vectors

We want to extend the idea of *perpendicular* to more than 2 dimensions.

Orthogonal Vectors

We want to extend the idea of *perpendicular* to more than 2 dimensions.

Definition

Vectors $\vec{\mathbf{u}}$ *and* $\vec{\mathbf{v}}$ *are* <u>*orthogonal*</u> *if* $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0$.

Orthogonal examples

Definition

Vectors $\vec{\mathbf{u}}$ *and* $\vec{\mathbf{v}}$ *are* *orthogonal* *if* $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0$.

Example

Let $\vec{\mathbf{u}} = \langle 3, -2 \rangle$ *and* $\vec{\mathbf{v}} = \langle 4, 6 \rangle$. *Then*

Orthogonal examples

Definition

Vectors $\vec{\mathbf{u}}$ *and* $\vec{\mathbf{v}}$ *are* *orthogonal* *if* $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0$.

Example

Let $\vec{\mathbf{u}} = \langle 3, -2 \rangle$ *and* $\vec{\mathbf{v}} = \langle 4, 6 \rangle$. *Then* $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 3(4) + (-2)(6) = 0$.
So $\vec{\mathbf{u}}$ *and* $\vec{\mathbf{v}}$ *are orthogonal.*

Orthogonal examples

Definition

Vectors $\vec{\mathbf{u}}$ *and* $\vec{\mathbf{v}}$ *are* *orthogonal* *if* $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0$.

Example

Let $\vec{\mathbf{u}} = \langle 3, -2 \rangle$ *and* $\vec{\mathbf{v}} = \langle 4, 6 \rangle$. *Then* $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 3(4) + (-2)(6) = 0$. *So* $\vec{\mathbf{u}}$ *and* $\vec{\mathbf{v}}$ *are orthogonal.*

Example

Similarly, $\vec{\mathbf{0}}$ *and any other vector are orthogonal, since* $\vec{\mathbf{0}} \cdot \vec{\mathbf{u}} = 0(u_1) + 0(u_2) + 0(u_3) = 0$.